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Anyons: Pseudo-integrability, Symmetry reduction and Semiclassical Spectrum

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Abstract

At the classical level anyons with harmonic confinement are known to exhibit two important properties namely partial separability and pseudo-integrability. These stem from the fact that this system is locally identical to isotropic oscillator system but differs in the global topology of the phase space. We clarify the meaning of pseudo-integrability and show that it amounts to a definite reduction of the symmetry group. We elaborate on the role of the fundamental group of the phase space and pseudo-intrgrability in the context of periodic orbit theory and obtain evidence of non-exactly known eigenvalues from the semiclassical trace formula. We also discuss an ambiguity regarding the 'half period' trajectories suggested by classical modeling and exhibited by the exactly known propagator for two anyons.

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I. INTRODUCTION

Anyons as a research area is now over twenty years old [1]. These systems have emerged as being interesting in their own right from mathematical physics point of view both at the classical and the quantum level. These systems constitute an example of inequivalent quantizations due to non trivial fundamental group of the configuration space [2]. These two dimensional systems have a configuration space whose fundamental group is a braid group [3]. Its one dimensional unitary representations give the kinematic classification labelled by the so called "statistical" parameter α. Being a system of finitely many degrees of freedom these are simpler to analyze for the dynamical consequences such as the spectrum of the Hamiltonian.

The quantum mechanical spectrum (with harmonic potential added) shows two distinct qualitative features: a) eigenvalues values which depend *linearly* on the statistical parameter, α, all of which are exactly known and b) eigenvalues which depend *non-linearly* on α and none of which is exactly known. These features were traced to the properties of partial separability and pseudo-integrability manifested at the classical level [4].

Partial separability of the Hamiltonian was exhibited explicitly in terms of two collective degrees of freedom and the remaining "relative" degrees of freedom. The total Hamiltonian, after removing the center of mass degrees, was obtained as a sum of $H_1(collective) + H_2(collective, relative)$. The commutator of H_1 and H_2 was shown to be proportional to H_2. This implies that the subspace of eigenstates states of the full H on which H_2 vanishes are exact eigenstates states of H_1 and these give *all* the exactly

known eigenvalues values and eigenstates states.

Pseudo-integrability, a concept first introduced by Richens and Berry [5], was described by exhibiting 2N constants of motion in involution which fail to lead to integrability via usual action-angle coordinates. This was conjectured to be the reason for the level repulsion seen numerically in the non-linearly interpolating spectrum [4]. The precise meaning of pseudo-integrability in this context, however, was not elaborated. In particular, how this classical feature translates into there being only two good quantum numbers, was not analysed.

There is another approach one could follow to get a handle on the many anyon spectrum namely the stationary phase approximation (SPA) to the propagator $G(E + i\epsilon)$ developed using a suitable path integral representation [6]. In this approach the propagator is typically obtained as a sum over (families of) periodic trajectories in the classical phase space. This is entirely given in terms of classical quantities. One therefore expects to see directly the effects of non-trivial fundamental group of the phase space, pseudo-integrability and partial separability in a semiclassical framework. Since at the classical level, anyons with oscillator confinement, are *locally* identical to the oscillator system (but differ in the global topology of the phase space), the periodic orbits are known. Thus application of the periodic orbit theory (POT) to anyons, may be expected to be tractable.

There is one point to be noted at the outset. At the quantum level, the statistical parameter, α_q, enters via the stipulation of multivaluedness of the wave functions and is *dimensionless*. At the classical level its counterpart, α_c, enters as the coefficient of a

total derivative term in the Lagrangian and has the dimensions of $\hbar$. These two must be related as $\alpha_c = \hbar\alpha_q$. In the SPA computation, α_c is held fixed with $\hbar$ going to zero. In the leading approximation, the spectrum will depend *linearly* on α_c and thus also on α_q. Alternatively, for comparison with the quantum spectrum to leading order, one will want to keep α_q fixed implying that α_c to go to zero with $\hbar$. Viewed either way, one may *not* expect to see *non-linear* dependence on the statistical parameter at the level of the leading approximation. Nonetheless, if the non-linearly interpolating eigenvalues also have a linear piece, one should be able to see it in the leading SPA.

In this work, we demonstrate that the classical analysis is adequate for exhibiting quantum symmetries. In this process we sharpen and clarify the meaning of pseudo-integrability and role of fundamental group of the phase space.

We also sketch application of the periodic orbit theory to many anyons. In the process we show the presence of eigenvalues which are potentially non-linearly interpolating. Such eigenvalues have been seen in the numerical spectrum for three and four anyons [7] and have been conjectured to be present for general N [8]. We obtain these systematically from POT.

Further, both the classical modeling and the exact propagator for two anyons, indicate an ambiguity regarding possibility of closed orbits with half the basic period and its inclusion in the trace formula. We discuss this ambiguity in some details.

The paper is organized as follows.

In section II we specify the classical model. The system is shown to be identical to just the isotropic oscillator *locally* but not at the global topological level. The set of classical trajectories is detailed.

In section III we classify the classical trajectories using symmetry transformations and explain how the non-trivial global topology reduces the symmetry group.

In section IV we sketch the POT application to anyons and demonstrate existence of eigenvalues other than the exactly known ones.

Section V contains a discussion of the issue of half period trajectories. We show that it is possible to regularise the point-like statistical flux and deduce the existence of half period trajectories. We look for their evidence in the exact propagator for two anyons and point out their ambiguous status.

Section VI contains general remarks on the role of the fundamental group, possible reduction of symmetries due to non-trivial topology of the phase space. A summary of results and conclusions is presented.

Appendix A contains details of $OSp(4N, R)$ symmetry.

Appendix B contains discussion of regularised classical dynamics of anyons and the presence of half period trajectories.

II. CLASSICAL MODELING AND DYNAMICAL TRAJECTORIES

Anyons are fundamentally defined as two dimensional quantum mechanical systems with a specified multi-valuedness for its wave function. The multi-valuedness is stipulated in terms of one dimensional non-trivial representations of the fundamental group of the configuration space, Q. At the least one defines the configuration space for N anyons to be,

$$Q \sim R^{2N} - \Delta, \tag{1}$$

where Δ is the set of points in the plane at which two or more particle positions coincide. Denoting by $\vec{r}_{ij}$ the difference of $\vec{r}_i$ and $\vec{r}_j$,

$$\Delta \equiv \{\vec{r}_i \ / \ \vec{r}_{ij} = \vec{0} \text{ for some } i \neq j\} \tag{2}$$

The fundamental group of Q is the so called pure braid group and is known to be non-trivial [3]. (For indistinguishable particles Q should be modded by the permutation group S_N. For most of what follows, this is not essential.) The multi-valued wave functions can be expressed in terms of single valued wave functions by a specific explicit multi-valued pre-factor while differentials of multi-valued wave functions can be expressed as "covariant" differentials of single valued wave functions. This is discussed in detail in reference [9]. Using the same notation as ref. [9], in terms of single valued wave functions, the quantum mechanical Hamiltonian for N-anyons with harmonic confinement is given by,

$$H = \frac{1}{2}\hbar\omega\{\sum_i(\vec{p}_i^2 + \vec{r}_i^2) - \alpha_q \sum_{i\neq j}\frac{\vec{r}_{ij}\times\vec{p}_{ij}}{r_{ij}^2} + \alpha_q^2 \sum_{i\neq j,k}\frac{\vec{r}_{ij}\cdot\vec{r}_{ik}}{r_{ij}^2 r_{ik}^2}\} \tag{3}$$

Here $\vec{p}_i \ \equiv \ -i\vec{\nabla}$. It is convenient to introduce the "statistical" gauge potential $\vec{A}_i$ as,

$$\vec{A}_i \equiv \alpha_q \sum_{j\neq i} \hat{k} \times \frac{\vec{r}_{ij}}{r_{ij}^2} \qquad (4)$$

where $\hat{k}$ is the unit vector in the z direction. Introducing $\vec{\pi}_i \equiv \vec{p}_i - \vec{A}_i$, the Hamiltonian can be expressed as ,

$$H = \frac{1}{2}\hbar\omega\{\sum_i(\vec{\pi}_i^2 + \vec{r}_i^2)\}. \qquad (5)$$

It is easy to see that the $\vec{\pi}_i$ and the $\vec{r}_i$ satisfy the same commutation relations as $\vec{p}_i$ and $\vec{r}_i$ away from the set Δ.

Remarks:

1) In the above quantum expressions $\vec{p}_i, \vec{r}_i, \alpha_q$ are all dimensionless. For the classical expressions below we will use α_c which has the dimensions of $\hbar$. These two are related by $\alpha_c \equiv \hbar\alpha_q$.

2) The use of $\vec{\pi}_i, \vec{r}_i$ coordinates is the classical counter part of the quantum "anyonic gauge" where the Hamiltonian operator expression looks simple and the wave functions are multi-valued.

The corresponding classical system is defined with Q as the configuration space and its cotangent bundle as the phase space (Γ). The Hamiltonian is the same as one given in the eq.(3) above. Noting that the Poisson Brackets (PB) among $\vec{p}_i, \vec{r}_i$ are the same as those among $\vec{\pi}_i, \vec{r}_i$ we see that going from $\vec{p}_i$ to $\vec{\pi}_i$ is a canonical transformation. The classical Hamiltonian may then be taken to be as given in eq.(5). *Locally* then the anyon system is identical to the 2N dimensional isotropic oscillator. The only difference is in the global topology of the phase space which turns out to have non-trivial consequence.

The Lagrangian from which the Hamiltonian, eq.(3), may be obtained is given by,

$$L = \frac{1}{2}\left\{\sum_i (\dot{\vec{r}}_i^2 - \vec{r}_i^2) + \alpha_c \sum_{i\neq j} \dot{\theta}_{ij} \ )\right\}, \qquad \theta_{ij} \equiv tan^{-1}(\frac{y_{ij}}{x_{ij}}). \tag{6}$$

The α_c dependent term being a total derivative implies that the equations of motion are identical to those of the oscillator. The class of solutions is of course *not* the same due to the removal of the set Δ. Note also that there is no explicit appearance of α_c either in the equations of motion or their solutions. Presence of α_c only changes the classical action and also affects the set of solutions qualitatively.

The configuration space equations of motion are:

$$\ddot{\vec{r}}_i \ = \ \vec{r}_i \ \ \forall i \quad \Rightarrow \ \ddot{\vec{r}}_{ij} \ = \ \vec{r}_{ij}. \tag{7}$$

Therefore in the plane defined by any two vectors $\vec{r}_i$ and $\vec{r}_j$ the difference vector traces an ellipse in general. For the oscillator degenerate ellipse (straight line) is permitted but not for the anyons.

Looking at the Hamiltonian form, one may expect to have the same symmetries for anyons as for the oscillator, namely the full OSp(4N,R) symmetry. Indeed rank of this group being 2N implies existence of 2N constants of motion in involution. These were exhibited in reference [4] and were stated to be indicative of the pseudo-integrability property. It was pointed out that integrability via action-angle coordinates requires further global input.

III. CLASSIFICATION OF ORBITS AND SYMMETRY REDUCTION

In this section we will carry out a classification of orbits for N anyons and see how the pseudo-integrability amounts to a reduction of the dynamical symmetry. We will

also see the role played by the fundamental group of the phase space in this regard. Noting that a trajectory is specified by giving an initial point, we will use the dynamical symmetry of the oscillator system to group various orbits into continuous families. For anyons, certain points and hence certain orbits are disallowed (removed) with the result that one gets several continuous families.

A. The case of two anyons

As usual we can separate the center-of-mass (CM) dynamics from the relative coordinate dynamics. The CM dynamics is trivial – identical to an oscillator locally and globally. The anyonic feature is contained entirely in the relative dynamics described as:

$$
\begin{aligned}
Q &= R^2 - \vec{0} \qquad &, \qquad \Gamma &= Q \times R^2 \\
L &= \tfrac{1}{2}(\dot{\vec{r}}^2 - \vec{r}^2) + \alpha_c \dot{\theta} \qquad &, \qquad \theta &\equiv tan^{-1} y/x \\
H &= \tfrac{1}{2}(\vec{\pi}^2 + \vec{r}^2) \qquad &, \qquad \vec{\pi} &\equiv \vec{p} - \alpha_c \tfrac{\hat{k}\times\vec{r}}{r^2}
\end{aligned}
\tag{8}
$$

The classical trajectories are trivially known in these cases. These are ellipses with two constants of motion, the energy (E) and the angular momentum ($J = (\vec{r} \times \dot{\vec{r}})_z$). The sign of J gives the sense of traversal and a trajectory reaches the point $\vec{r} = \vec{0}$ if and only if $J = 0$. These degenerate trajectories ($J = 0$) are not allowed for the anyons though are allowed for the oscillator.

Consider the set of all possible trajectories with a given energy. Fix two such trajectories with angular momenta J, J'. Suppose there exists a continuous family of interpolating trajectories connecting these two. Clearly if the angular momenta have opposite signs, then at least one of the interpolating trajectories must be degenerate. But this is disallowed for anyons. Hence, for anyons there must be *at least* TWO 'orientation'

classes of trajectories distinguished only by the sign of angular momenta.

While we have shown that there must be at least two families we do not know yet if there *must* be *precisely* two families. We will use the symmetries to show that there are precisely two families for anyons and precisely one family for the oscillator.

As far as the PB algebra is concerned, the oscillator and anyons are identical and hence we do have *infinitesimal* symmetries forming the Lie algebra of $U(2)$ (or $OSp(4,R)$) for both cases. The constant H surface (S^3 in the phase space R^4 for oscillator) is of course invariant under the infinitesimal symmetry transformations. For anyons, the phase space has $\vec{r} = \vec{0}$ removed and the constant H surface has a great circle (one dimensional) removed from the S^3, say a "longitude".

In the appendix A we have given the details of the $OSp(4N,R)$ symmetry of the oscillator. In the present case of two anyons with center of mass coordinate separated, we have effectively, $N = 1$ and the symmetry generators form $OSp(4,R)$. We have only the 4 generators of the $\mathbf{T}_i$ type. Of these four, $\mathbf{u}_1$ generates rotations of position and momentum while $\mathbf{u}_2$ generates time evolution. The generic integral curves are given in the appendix A.

From appendix A we also recall that $OSp(4,R)$ acts transitively on $H = E$ sphere. This shows that for the oscillator there is precisely ONE family of trajectories. For anyons we need analogous result with the extra condition that degenerate trajectories are avoided to conclude that there are precisely TWO families.

Claim Given any two trajectories with the same sign of their angular momenta, there exist a one parameter group connecting the two without changing the sign of angular momenta.

Proof Observe that $\mathbf{u}_1, \mathbf{u}_2$ transformations leave the angular momentum invariant. Choose $\mathbf{u} = cos(\beta)\mathbf{u}_3 + sin(\beta)\mathbf{u}_4$. The one parameter group generated by this generator transforms the angular momentum as, (appendix A)

$$2J(\sigma) \;=\; 2J - sin^2(\sigma)\left\{2J - \bar{\omega}\bar{\mathbf{u}}\mathbf{L}\mathbf{u}\omega\right\} + sin(2\sigma)\left\{\bar{\omega}\mathbf{L}\mathbf{u}\omega\right\} \tag{9}$$

For the choice of $\mathbf{u}$ made, $\bar{\mathbf{u}}\mathbf{L}\mathbf{u} = -\mathbf{L}$ and putting $J_u \equiv \frac{\bar{\omega}\mathbf{L}\mathbf{u}\omega}{2}$,

$$\begin{aligned} J(\sigma) &= J\ cos(2\sigma) + sin(2\sigma)\ J_u \\ &= \sqrt{J^2 + J_u^2}\ cos(2\sigma - \delta)\ , \qquad \delta \equiv tan^{-1}(J_u/J) \in (-\pi/2, \pi/2) \end{aligned} \tag{10}$$

This is valid for all choices of β. Observe that J_u depends on β. Furthermore it satisfies,

$$\begin{aligned} \frac{\partial^2 J_u(\beta)}{\partial\beta^2} &= -J_u(\beta) && \Rightarrow \\ J_u(\beta) &= J_u(0)cos(\beta) + \frac{\partial J_u}{\partial\beta}(0)sin(\beta) && \Rightarrow \\ (J_u)_{max} &= \left\{ (J_u(0))^2 + (\frac{\partial J_u}{\partial\beta}(0))^2 \right\}^{1/2} && = \sqrt{E^2 - J^2} \end{aligned} \tag{11}$$

The last equality follows by explicit evaluation. Choosing β to maximise J_u then implies that the square root in the equation for $J(\sigma)$ is just the energy E. Since for any given energy, we must have $J^2 \leq E^2$, it follows that $J(\sigma)$ covers the entire range of possible J values.

From this it follows that we can always choose a $\hat{\sigma}$ such that $J(\hat{\sigma}) = J'$ without passing through zero angular momentum. This proves the claim. Hence there are precisely TWO families of trajectories each spanning the whole orientation class. This gives the classification of trajectories.

One may view the S^3 as compactification of R^3 with the south pole as the origin of R^3 and the north pole as the point at infinity. The removal of Δ now corresponds to removal of an infinite line, say, the z-axis of the R^3. It is immediate that the fundamental group is the group of integers, Z, and therefore the basic trajectories winding around the z-axis belong to two orientation classes. In this case the fundamental group directly shows how the basic trajectories are naturally classified into the "orientation classes" .

Consider now the Hamiltonian vector fields generating the three one parameter subgroups of $SU(2), \mathbf{u}_1$ generates rotations about the z-axis. Its orbit through any generic point avoids the z-axis and the vector field remains complete (its integral curves range over the full real line) . The other two subgroups on the other hand have orbits *necessarily* cutting through the z-axis. These vector fields therefore are necessarily incomplete (for the anyon case). The transformations generated by these vector fields do *not* for a group and the dynamical symmetry group for anyons is reduced from $U(2)$ to $U(1) \times U(1)$.

Thus the removal of the set Δ has altered the fundamental group, has classified trajectories into orientation classes and has also reduced the dynamical symmetry group form $U(2)$ to $U(1) \times U(1)$ due to the incompleteness of vector fields.

Note that while the symmetry group is reduces from $U(2)$ to $U(1) \times U(1)$ for anyons, the rank has remained the same and therefore the anyonic system continues to be integrable. The integral curves being circles, we have integrability via action-angle variables and we get the exact spectrum from the semiclassical approximation. In fact one can follows the EBK quantization procedure to reproduce the exact spectrum. [13].

B. The case of many anyons

The $N > 2$ case differs from the $N = 2$ case in many ways. The potential symmetry group, $OSP(4N, R)$, (see appendix A) generating families of trajectories is lot more complicated and so is the fundamental group for the phase space of anyons. However the analysis of the previous section already suggests a strategy for obtaining a classification of the families of trajectories. A generic trajectory may be viewed as a collection of $\frac{N(N-1)}{2}$ ellipses traced by the $\vec{r}_{ij}, i < j$. To get a convenient handle on the disallowed trajectories, define $J_{ij} \equiv (\vec{r}_{ij} \times \dot{\vec{r}}_{ij})_z$. Clearly a trajectory will cut through the set Δ if and only if J_{ij} vanishes for at least one pair of indices. We will term such a trajectory as degenerate. For anyon only non-degenerate trajectories are allowed. Note that J_{ij} are constants of motion just as J_i and $E_i \equiv \frac{1}{2}(r_i^2 + p_i^2)$ [4] are. Each of these elliptical trajectories will have a sense of traversal ($\epsilon_{ij} \equiv$ sign of J_{ij}). Thus a basic trajectory of N particles has an associated set: $\{\epsilon_{ij}\}$. Exactly as in the previous sub-section, the set of non-degenerate trajectories can be classified by these 'orientations'. Provided we can exhibit trajectories which realise all possible choices of ϵ_{ij}, we will have *at least* $2^{N(N-1)/2}$ families for anyons and precisely as many if every member of a class is connected to every other member of the same class by a symmetry transformation without crossing Δ. It is easy to show that every choice of ϵ_{ij} is realised by considering concentric circular trajectories for each of the particles with various possible signs for the angular momenta

J_i and various possible ordering of the radii of the circular trajectories. For oscillator we will have precisely one family if $OSp(4N, R)$ acts transitively on the constant energy sphere, S^{4N-1}.

The proof of transitivity in the case of oscillator is most directly obtained by noting that $OSp(4N, R)$ is isomorphic to $U(2N, C)$ and it is well known (and easy to see) that the coset space $U(2N, C)/U(2N-1, C)$ is diffeomorphic to S^{4N-1}. This immediately shows that for the oscillator, there is precisely ONE family of periodic trajectories. The counting of families is more tricky in the case of the anyons. Let us see the symmetry reduction in a slightly different manner.

Let Δ' denote the set of all degenerate trajectories i.e. at least one of the J_{ij}'s is zero. Note that Δ and Δ' are different, the former being a proper subset of the latter. Suppose we find the subgroup of symmetries which leave Δ' invariant, then the same subgroup will also leave the set of all non-degenerate trajectories invariant. Since we are looking for a group, we can consider the infinitesimal action. Let $J \equiv \prod_{i<j} J_{ij}$. Clearly, $J = 0$ characterises the set Δ'. If we have a degenerate trajectory with two or more J_{ij}'s zero, then every infinitesimal action will keep within the set of degenerate trajectories. If only one J_{mn} is zero then infinitesimal action will act invariantly if and only if J_{mn} is invariant. Of course we could have degenerate trajectories with different angular momenta being zero and hence for the invariance of Δ' it is necessary that the generators of the subgroup must leave each of the J_{ij} 's invariant *whenever* J_{ij} *itself is zero.* This subgroup is identified in the appendix A. There are two forms of generators. Any generator with all the block matrices being the same $\mathbf{u}$ matrix with $\mathbf{u} \in OSp(4, R)$ will leave all the J's invariant. These transformations act on the center-of-mass variable

alone and constitute the expected $OSp(4,R)$ symmetry. In addition, generators with only diagonal blocks being the same $\mathbf{u}$ and all the off-diagonal blocks being $\mathbf{0}$ also leaves the J's invariant provided $\mathbf{u} = \mathbf{u_1}$ or $\mathbf{u}_2$ (or a linear combination thereof). These are just the total Hamiltonian and the total angular momentum $(\sum_i J_i)$ which generate time evolutions and common rotations of all the positions and momenta. These are the only subgroups leaving the set of degenerate (and non-degenerate) trajectories invariant. The vector fields corresponding only to these will remain complete. Thus we see that the symmetry is reduced to $OSp(4,R) \times O(2) \times O(2)$. This is identical to the symmetry in the case of two anyons.

It is apparent from the above discussion that pseudo-integrability of anyons really means that not all the Lie algebra level symmetries exponetiate to Lie group symmetries. One should distinguish between "infintesimal integrability" and integrability. A general discussion is given in the last section.

This purely classical analysis has already explained the qualitative result that for anyons only the total energy and the total angular momentum are the conserved quantities (good quantum numbers). Since we also have a classification of periodic trajectories we will take the next step of attempting semiclassical approximation to see what further information can be obtained.

IV. SEMICLASSICAL SPECTRUM FOR ANYONS

The central quantity of interest is the propagator defined as,

$$
\begin{aligned}
G(E+i\epsilon) &\equiv Tr(\frac{1}{E-H+i\epsilon}) \\
&= \sum_n (E-E_n+i\epsilon)^{-1} \\
&= \sum_n \ \mathcal{P}((E-E_n)^{-1}) - i\pi \sum_n \ \delta(E-E_n)
\end{aligned} \tag{12}
$$

In the last equality, $\mathcal{P}$ denotes the principle value while the second term is of course the density of states. Defining $\tilde{G} \equiv iG$ allows us to write the density of states, $g(E)$ as ,

$$
\begin{aligned}
g(E) &= \frac{1}{2\pi} \ \{\tilde{G}(E+i\epsilon) + \tilde{G}^*(E-i\epsilon) \ \} \\
&= \frac{1}{\pi} \ Re(\ \tilde{G}(E+i\epsilon) \ )
\end{aligned} \tag{13}
$$

The derivation of the semiclassical trace formula begins with the definition of the propagator, $G(E)$, with trace operation expressed via a path integral representation. The trace operation combined with the stationary phase approximation (SPA) gives the propagator as a sum over periodic orbits (in the phase space), of terms with an amplitude given by a suitable Van-Vleck determinant times a phase whose exponent is the value of the 'action', $\int \mathbf{p} \cdot \mathbf{dq}$ around the periodic orbit together with a contribution from the Maslov indices. This works very well for isolated periodic orbits. However for systems with symmetries, as for the oscillator or anyons, the periodic orbits come in continuous families and a modification is needed.

In such cases, the sum over orbits gets replaced by a sum over families of orbits together with a measure factor. If the family arises due to a symmetry, as is usually the case, the amplitude and the phase is same for all members of the family and as such can be computed from any one member. This is discussed by Littlejohn et al [10] in detail. The origin of families of periodic orbits in the case of oscillator is of course the $OSp(4, R)$ dynamical symmetry and for anyons essentially the same transformations

generate families of trajectories. This is already discussed in the previous section.

To obtain the action along a periodic orbit one point needs to be noted. Observe that in terms of the $\vec{\pi}_i, \vec{r}_i$ coordinates there is no explicit reference to α and Δ is also precisely the set on which one gets δ-function singular PBs with α appearing as the coefficient. If one did inverse Legendre transformation to get a Lagrangian one would not get the α dependent term, but of course the configuration space will have the Δ removed. Using $\vec{p}_i, \vec{r}_i$ variables and then doing inverse Legendre transformation will produce the Lagrangian with the α dependent total derivative term. Now the removal of Δ is made explicit by the α dependent term. It is this explicit form which is convenient for computation of action for periodic orbits.

The action for the oscillator is equal to $(2\pi E)/\omega$ and the action for anyons is the same as that for the oscillator except for the contribution from the total derivative term in the Lagrangian. Its value is $\pm 2\pi\alpha$ with sign depending on the orientation class(es). We have already seen that for $N = 2$ we have precisely 2 orientation classes while for general N we have *at least* $2^{N(N-1)/2}$ classes.

Observe that *even without* the knowledge of precise number of families, we have precisely $2^{N(N-1)/2}$ sums in the sum over periodic trajectories. This is because, the contribution of the action integral depends only on the orientation class and not on further possible subclasses of trajectories. The measure factors therefore add up within each orientation class and this is sufficient to get the semiclassical eigenvalues.

The sum over families in the trace formula contains as many terms. To get the eigen-

values we need to look at only the action integral which differs from the oscillator only by the additional $\oint \frac{\alpha_c}{2} \sum_{i\neq j} \dot{\theta}_{ij}$ contribution. The oscillator's contribution to the action integral is of course just $\frac{2\pi}{\omega}E$, where E is the classical energy and ω is the oscillator frequency.

The net α_c dependent contribution to the action integral is given by,

$$\frac{\alpha_c}{2} \oint \sum_{i\neq j} \dot{\theta}_{ij} \ = \ 2\pi\alpha_c \sum_{i<j} \epsilon_{ij} \tag{14}$$

Here ϵ_{ij} is the sign of traversal of $\vec{r}_{ij}$ and is ± 1. The $\sum_{i<j}$ is trivial to evaluate.

Since the classical trajectories are identical to those of the oscillator, we will have the same Maslov indices which incorporate the usual zero point energy namely, $N(N-1)$ for $2N$ dimensional oscillator. Including this contribution, the semiclassical energies are given by :

$$E_n\{\epsilon_i\} \ = \ \hbar\omega(n \ + \ \alpha_q \sum_{i<j} \epsilon_{ij} \ + \ N(N-1) \ ) \qquad\qquad n \ \geq 0 \tag{15}$$

The coefficient of α_q takes all possible integer values from $-N(N-1)/2$ to $N(N-1)/2$ in steps of 2. The extreme values correspond to all the particles traversing the same way and represent the contribution of "collective" motion shown in [4]. The remaining values represent contributions of "relative" motion. These are new eigenvalues albeit at the leading $\hbar$ level. Numerical spectra available for $N = 3, 4$ show eigenvalues matching with the above at the $\alpha_q \ = \ 0$ and $\alpha_q \ = \ 1$ [7]. These numerical eigenvalues however are non-linearly interpolating while our semiclassical eigenvalues are linearly interpolating. It is possible that the semiclassical eigenvalues will receive corrections to make them non-linearly interpolating. In reference [8] also it was conjectured that there will be linearly interpolating eigenvalues with various slopes based on a different

semiclassical argument. We have obtained these eigenvalues by direct application of the periodic orbit theory and proved their existence for all N.

Observe that the α_q dependence in the eigenvalues (locations of poles of $G(E)$) arises only from the explicit $\alpha_c \equiv \alpha_q \hbar$ in the action. This therefore must be at the most linear in α_q. Since α_q is fixed, the semiclassical limit of $\hbar \to 0$ implies $\alpha_c \to 0$. Thus one may at the most see linear α_c dependence in the leading approximation which could however be indicative of exact non-linearly interpolating eigenvalues.

Thus the leading level semiclassical propagator already indicates the presence of eigenvalues such that $E(\alpha_q = 1) - E(\alpha_q = 0)$ takes all possible integer values between $\pm N(N-1)/2$ in steps of 2.

V. HALF PERIOD TRAJECTORIES AND EXACT PROPAGATOR FOR TWO ANYONS.

Recall that anyons are fundamentally defined quantum mechanically and all the energy eigenfunctions of the system vanish on the set Δ of coincident points. We modeled the classical system by removing Δ and explored the consequences in the previous sections. In particular we just *omitted* trajectories that could cut through Δ.

From a purely classical point of view, though, this is little unsatisfactory. One could legitimately ask just what happens to trajectories (eg $J = 0$ for $N = 2$) that attempt reaching the disallowed region? To decide this one has to *extend* the classical modeling by supplementing the equations of motions with a specified "boundary" condition. The

choice is helped by noting that the curl of the vector potential is a δ-function which may be "regulated" to stipulate the "boundary condition". This analysis is given in appendix B.

The result is that *all the elliptical trajectories are identical to those of the oscillator and only the degenerate trajectories get modified. A degenerate trajectory must reflect back from the coincident point.*

Clearly such an trajectory will have half the period of the generic trajectories and hence we will refer to these as the "half trajectories".

Should these orbits be included in the sum over orbits in the context of semiclassical approximation? Are they already "visible" in the exact spectrum known for two anyons? To answer these, let us look at the exact propagator for the (relative coordinate part of) the two anyon system.

The propagator can be computed exactly from the known exact spectrum [11]. The exact spectrum is given by,

$$E_{n,j} \;=\; \hbar\omega(2n + |j - \alpha_q| + 1), \quad n \geq 0, \quad j \in Z \tag{16}$$

From this the partition function is obtained as,

$$Z(\beta) \;=\; \frac{cosh(\ \beta\hbar\omega\ (\alpha_q - 1)\ ) + cosh(\ \beta\hbar\omega\alpha_q\ )}{2sinh^2(\beta\hbar\omega)} \tag{17}$$

and the exact density of states is obtained as:

$$g(E) \equiv \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} e^{\beta E} Z(\beta)$$

$$= \frac{E}{(\hbar\omega)^2}\{1 + \sum_{k\geq 1}\{cos(\frac{2\pi kE}{\hbar\omega} + 2\pi k\alpha_q) + cos(\frac{2\pi kE}{\hbar\omega} - 2\pi k\alpha_q)\}$$
$$-\frac{1}{(\hbar\omega)}\sum_{k\geq 1}\{2\alpha_q \ sin(2\pi k\alpha_q) \ sin(\frac{2\pi kE}{\hbar\omega})\} \tag{18}$$
$$+\frac{1}{(\hbar\omega)}\sum_{k\geq 1}\{(-1)^k \ sin(\pi k\alpha_q) \ sin(\frac{\pi kE}{\hbar\omega})\}$$

Rewriting the products of sines as differences of cosines we get,

$$\begin{aligned} g(E) = {} & \frac{E}{(\hbar\omega)^2} \\ & +\frac{1}{\hbar\omega}\ (\ \frac{E}{\hbar\omega} + \alpha_q\ )\sum_{k\geq 1}\ cos(\frac{2\pi kE}{\hbar\omega} + 2\pi k\alpha_q) \\ & +\frac{1}{\hbar\omega}\ (\ \frac{E}{\hbar\omega} - \alpha_q\ )\sum_{k\geq 1}\ cos(\frac{2\pi kE}{\hbar\omega} - 2\pi k\alpha_q) \\ & -\frac{1}{2\hbar\omega}\ \sum_{k\geq 1}\ (-1)^k\ cos(\frac{\pi kE}{\hbar\omega} + \pi k\alpha_q) \\ & +\frac{1}{2\hbar\omega}\ \sum_{k\geq 1}\ (-1)^k\ cos(\frac{\pi kE}{\hbar\omega} - \pi k\alpha_q) \end{aligned} \tag{19}$$

Remarks:

1. The partition function is manifestly invariant under $\alpha_q \to 1-\alpha_q$. The convergence of integrals in computing the density of states for $\alpha_q \in (0,1)$ requires $\frac{E}{\hbar\omega}$ to be greater than or equal to 1.

2. The first term above is the usual Thomas-Fermi term [11]. This will be suppressed in expressions below.

3. In the limit $\hbar \to 0$ keeping E, α_q fixed, the exact propagator effectively looses all dependence on α_q. This is also the semiclassical trace formula for the oscillator showing that the semiclassical approximation is exact for the oscillator. All the other terms are sub-leading.

4. In the semiclassical computation, one will use $\alpha_c = \hbar\alpha_q$ and the SPA will be done keeping α_c fixed. Then only the last two terms will be sub-leading and the propagator

will continue to have α_c dependence.

5. In the last two sub-leading terms the argument of cosine is half of that in second and the third term. As is well known, the argument of cosines is the classical action, $\int \vec{p}\cdot d\vec{r}$, around a classical orbit including possible contributions from "Maslov indices". The last two terms are therefore suggestive of contribution from "half orbits". Indeed if one takes a Fourier transform of $g(E)$ [11] , then one sees a peak at 1/2 period from the last two terms. *One may therefore conclude that half orbits should also be included in the trace formula.*

This turns out to be somewhat ambiguous. To see this let us express the propagator in a sum-over-orbits form. To do this rewrite the the last two terms to resemble the second and the third terms.

The last two terms also have a $(-1)^k$ in the summation over k and this can be handled in two ways.

(A) Separate these sums into even and odd integer sums. All the sums over k being geometric series can be done explicitly ($i\epsilon$ needs to be added). Defining $\mathcal{E} = E/(\hbar\omega)$, we get,

$$\begin{aligned}\hbar\omega g(E) = \mathcal{E}\ +& \\ &\frac{\frac{1}{2}\ (\mathcal{E}+\alpha_q-\frac{1}{2}\ )e^{2\pi i(\mathcal{E}+\alpha_q)}+\frac{1}{4}e^{i\pi(\mathcal{E}+\alpha_q)}}{1-e^{2\pi i\alpha_q(\mathcal{E}+\alpha_q)}}\ +\ \text{C. C.} \\ &\frac{\frac{1}{2}\ (\mathcal{E}-\alpha_q+\frac{1}{2}\ )e^{2\pi i(\mathcal{E}-\alpha_q)}-\frac{1}{4}e^{i\pi(\mathcal{E}-\alpha_q)}}{1-e^{2\pi i\alpha_q(\mathcal{E}-\alpha_q)}}\ +\ \text{C. C.}\end{aligned} \tag{20}$$

Here C. C. means complex conjugation. Now we can "read-off" $G(E+i\epsilon)$ and get (suppressing the Thomas-Fermi term),

$$(-2\pi i)^{-1}\hbar\omega G(E+i\epsilon) = \frac{1}{2}\ \{\mathcal{E}+\alpha_q - \frac{1}{2}(1-e^{-i\pi(\mathcal{E}+\alpha_q+i\epsilon)})\ \}\ \ \{\frac{e^{2\pi i(\mathcal{E}+\alpha_q+i\epsilon)}}{1-e^{2\pi i(\mathcal{E}+\alpha_q+i\epsilon)}}\} + \frac{1}{2}\ \{\mathcal{E}-\alpha_q + \frac{1}{2}(1-e^{-i\pi(\mathcal{E}-\alpha_q+i\epsilon)})\ \}\ \ \{\frac{e^{2\pi i(\mathcal{E}-\alpha_q+i\epsilon)}}{1-e^{2\pi i(\mathcal{E}-\alpha_q+i\epsilon)}}\} \quad (21)$$

Noting that the second group of braces is sum of a geometric series:

$$\frac{e^{2\pi i(\mathcal{E}\pm\alpha_q+i\epsilon)}}{1-e^{2\pi i(\mathcal{E}\pm\alpha_q+i\epsilon)}} = \sum_{k\geq 1}\ e^{2\pi i k(\frac{E}{\hbar\omega}\pm\alpha_q+i\epsilon)} \quad (22)$$

we see that the propagator *is* expressed in a form suggestive of a sum over periodic orbits. The first group within the braces being the "amplitude" while the second group being contributions from multiple traversals of a basic periodic orbit. The two terms can be seen to come from families of elliptical orbits going clockwise and anti-clockwise sense and that there *no* term corresponding to degenerate ellipses or half orbits. Note that the terms in the "amplitudes" other than $\mathcal{E}$ are sub-leading relative to $\mathcal{E}$. This leading part of the "amplitudes" can also be seen to come from POT in presence of continuous families of periodic orbits [10]. The "amplitudes" however are complex. For $\alpha_q = 0$ there is cancellation of these sub-leading terms and one recovers the exact result for the oscillator. It appears that though the system is integrable, the trace formula does *not* give exact propagator.

The poles in $G(E)$ come from the two terms and are given by:

$$E_+(n_+) = \hbar\omega(n_+ - \alpha_q) \qquad n_+\ \geq 2 \quad (23)$$

$$E_-(n_-) = \hbar\omega(n_- + \alpha_q) \qquad n_-\ \geq 1 \quad (24)$$

The residues at these poles ($\mathcal{E}\pm\alpha_q = n_\pm$) are $n_\pm$ if $n_\pm$ is *even* and $n_\pm - 1$ if $n_\pm$ is *odd*. These residues of course give the degeneracy. *Note that the sub-leading terms are important for this.*

It is easy to see that the locations of all the poles can be re-expressed in the form given by the exact spectrum. The degeneracies at the poles also match exactly as expected.

(B) We can also write $(-1)^k cos(k\theta) = cos(k(\theta \pm \pi))$. Proceeding exactly as in the case (A), we can "read-off" $G(E + i\epsilon)$ and get,

$$\left(\frac{-\hbar\omega}{2\pi i}\right) G(E+i\epsilon) = \frac{1}{2}\{\mathcal{E}+\alpha_q\}\{\frac{e^{2\pi i(\mathcal{E}+\alpha_q+i\epsilon)}}{1-e^{2\pi i(\mathcal{E}+\alpha_q+i\epsilon)}}\} \; + \; \frac{1}{2}\{\mathcal{E}-\alpha_q\}\{\frac{e^{2\pi i(\mathcal{E}-\alpha_q+i\epsilon)}}{1-e^{2\pi i(\mathcal{E}-\alpha_q+i\epsilon)}}\}$$
$$-\frac{1}{4}\;\{\frac{e^{\pi i(\mathcal{E}-1+\alpha_q+i\epsilon)}}{1-e^{\pi i(\mathcal{E}-1+\alpha_q+i\epsilon)}}\} \quad + \quad \frac{1}{4}\;\{\frac{e^{\pi i(\mathcal{E}+1-\alpha_q+i\epsilon)}}{1-e^{\pi i(\mathcal{E}+1-\alpha_q+i\epsilon)}}\} \qquad (25)$$

These have poles which are subsumed by the poles from the first two terms. The overall pole structure and residues of course are exactly same as before as they should be.

The exponents in the last two terms however do *not* look like the action integral. A reflecting trajectory will not receive a contribution from the α_q dependent total derivative term. The reflecting trajectories will also form a single family of trajectories and thus should give only one term and not two terms. The last two terms can be combined and the propagator may be re-expressed as,

$$\left(\frac{-\hbar\omega}{2\pi i}\right) G(E+i\epsilon) = \sum_{k\geq 1} \; \Big[\; (\frac{\mathcal{E}+\alpha_q}{2}) \; e^{2\pi ik(\mathcal{E}+\alpha_q+i\epsilon)} \; + \; (\frac{\mathcal{E}-\alpha_q}{2}) \; e^{2\pi ik(\mathcal{E}-\alpha_q+i\epsilon)}$$
$$+ \; (\frac{-i \; sin(k\pi\alpha)}{2}) \; e^{i\pi k(\mathcal{E}-1)} \; \Big] \qquad (26)$$

The propagator now does have form indicating a contribution from the half period orbits including a contribution from a Maslov index due to reflection. However it has a complex amplitude with a dependence on α_c and the multiple traversals index k.

Thus the exact propagator can be expressed in two different forms mimicking the trace formula with and without the half period trajectories. Although the phases are as expected, the amplitudes are not. These terms also have different orders of $\hbar$.

The different orders of $\hbar$ seen can be understood in the context of trace formula as due to *families* of periodic trajectories and not due to any higher order corrections to the SPA. When periodic trajectories come in continuous families, the number of Gaussian integrations is reduced since the integration over the family must be done exactly [10] and this increases the powers of $\hbar$ in the denominator. The elliptic trajectories form a three parameter family while the half trajectories form a one parameter family which explains the difference in the powers of $\hbar$.

While the powers of $\hbar$ can be understood, the precise matching of the amplitudes from the POT does not seem to follow. The appearance of complex amplitudes may be indicative of the need for complex trajectories and/or precise matching of semiclassical wave functions across the excluded regions. This is beyond the scope of present work.

The ambiguous "presence" of half trajectories seems to suggest that one may *either* exclude these altogether but extend the trace formula to deal with the non-trivial topology due to the excluded regions *or* include these trajectories as additional families but perhaps use complex trajectories. Needless to say that the structure of half trajectories

will be more complicated for $N > 2$. For getting the semiclassical eigenvalues, the first alternative may suffice and is already adequate to indicate the presence of non-exactly known eigenvalues. As there does not appear to be any scope for going beyond the leading SPA (all intermediate exponents of phases are quadratic), it is not clear how one may develop a semiclassical scheme for obtaining the non-linear α_q dependence.

VI. SUMMARY AND CONCLUSIONS

In the previous sections we encountered two important features, one related to the fundamental group and one related to symmetry reduction due to incomplete vector field. Both were caused by the same source, namely removal of coincident points implying topologically non-trivial phase space. Both are relevant for periodic orbit theory in a general way and a few general remarks are in order.

The stationary phase approximation to the propagator naturally leads to periodic orbits in the phase space. These orbits may be isolated or come in continuous families or both. The continuous families may be generated by a full group of symmetries or by only a subset of symmetry transformations. Since each periodic orbit is also a map of S^1 to the phase space Γ, clearly every orbit must belong to one and only one homotopy class of the fundamental group, $\pi_1(\Gamma)$, of the phase space. If an orbit is a member of a continuous family then the entire family must belong to a single homotopy class. Note that a given homotopy class may contain no orbit (solution of equation of motion) or several isolated orbits and/or several families of orbits. But a family can not spill over two distinct homotopy classes. In section III we saw precisely the splitting of a single basic family for oscillator into two basic families for anyons because of the non-trivial

fundamental group. Multiple traversals of course belong to different homotopy classes and are explicitly summed over. Thus a non-trivial π_1 may (but not necessarily) provide an obstruction to a symmetry. How exactly may such an obstruction manifest itself? For this we have to consider vector fields generating symmetries.

Recall [12] that every function on the phase space generates *infinitesimal* symplectic diffeomorphisms (canonical transformations) via its corresponding (globally) Hamiltonian vector field. The Lie algebra of such vector field is isomorphic to the Possion Bracket (PB) algebra of functions on Γ. However such infinitesimal transformations exponentiate to give a one parameter group of transformations *only if* the vector field is *complete* i.e. the integral curves of the vector field can be extended so as to have these as a map from the full R. Only complete vector fields - and this is a global statement - give rise to groups of symmetries. (A corresponding quantum mechanical statement for continuous symmetries is contained in the Stone's theorem: every one parameter group of unitary transformations is generated by a self-adjoint operator and conversely.) Since we usually want classical symmetries to be reflected at the quantum level with observable generators we have to have *groups* of symmetries and hence complete vector fields.

The criterion of integrability in terms of vanishing PB's guarantees only the existence of *infinitesimal symmetries* which is of course a prerequisite. An "infinitesimally integrable" system may thus be: (a) integrable via action-angle coordinates if the vector fields are complete and the integral curves are periodic; (b) integrable, but not by action-angle variables if the vector fields are complete but only a subset of these have periodic integral curves and (c) partially integrable if only a subset of vector fields are

complete. The “pseudo integrability” property of anyons pointed out in ref. [4] falls in the category (c). As explained in the appendix A apart from the Hamiltonian only the total angular momentum has a complete vector field and hence is the only quantum number that survives.

A non-trivial fundamental group by itself however does not imply possibility of incomplete vector fields. For, on a compact manifold all vector fields are complete and it can of course have a non-trivial π_1. In many cases constant energy surfaces are compact and one does not have to worry about incomplete vector fields. For the anyons however removal of the set Δ makes the constant energy surface noncompact which admits possibility of incomplete vector fields and corresponding loss of symmetry.

In the present work we have seen manifestation of all these features.

We also considered application of the periodic orbit theory to this locally trivial (integrable) but globally non-trivial system. As a by-product, we saw how the exact spectrum for two anyons is reproduced by semiclassical methods. We saw that while the dynamical symmetry is reduced from $SU(2)$ to $U(1)$, the rank remained the same and hence integrability property is preserved.

For $N \geq 3$, we reproduced the previously known exact eigenvalues [4]. In addition, we obtained further *new* linearly interpolating eigenvalues. In the language of reference [4], these correspond to effect of ‘relative’ dynamics. In this case the symmetry reduction was drastic, from $OSp(4N, R)$ to $OSp(4, R) \times O(2, R) \times O(2, R)$. The rank was reduced from $2N$ to 6, destroying the integrability property.

We discussed in detail the issue of "half" trajectories and pointed out their ambiguous role. Taking the view-point that "half" trajectories be excluded, and noting that there does not seem to be any scope for computing higher order corrections to the semiclassical spectrum with a possible non-linear dependence on α_q, it seems that the non-linearly interpolating eigenvalues are genuine quantum consequences beyond what semiclassical analysis could give. Semi-classical analysis is nevertheless sufficient to indicate the presence of these eigenvalues.

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Appendix A : **OSp dynamical symmetry and classification of trajectories**

In this appendix we collect together some of the well known facts about the dynamical symmetry of the n dimensional isotropic oscillator.

The isotropic oscillator in n dimensions has $OSp(2n, R)$ as the group of dynamical symmetries. This is a group of $2n \times 2n$ order matrices, $\mathbf{g}$, which are both orthogonal and symplectic. Denoting by $\bar{\mathbf{g}}$ the transpose of $\mathbf{g}$, we have the defining equations:

$$
\begin{array}{rlcll}
 & \bar{\mathbf{g}}\,\mathbf{g} & = & \mathbf{I}_{2n} & : \text{ (orthogonal)} \\
 & \bar{\mathbf{g}}\,\mathbf{\Omega}\,\mathbf{g} & = & \mathbf{\Omega} & : \text{ (symplectic)} \\
\text{and with} & \mathbf{g} & \approx & \mathbf{I}_{2n} + \epsilon\mathbf{T} & \text{generators } T \text{ satisfy} \\
 & \bar{\mathbf{T}} & = & -\mathbf{T} & \\
 & \bar{\mathbf{T}}\,\mathbf{\Omega} & = & -\mathbf{\Omega}\,\mathbf{T} &
\end{array}
\tag{27}
$$

Here $\mathbf{I}_{2n}$ is the identity matrix of order $2n$ while $\mathbf{\Omega}$ is a suitable matrix defining the symplectic condition. This will be chosen below.

The symplectic condition ensures that we have a (linear) canonical transformation while orthogonality ensures that the Hamiltonian, $\sum_i (p_i^2 + q_i^2)/2$, is invariant. It is easy to see that dimension of this group is n^2 while its rank is n. In fact one can show that the group $OSp(2n, R)$ is isomorphic to the group $U(n, C)$. Making an explicit choice of Ω in a block form, one can obtain the block form for $\mathbf{g}$ using the symplectic condition. The real block matrices can be combined into complex block matrices. The orthogonality condition in terms of real matrices then translates into the unitarity condition for the complex matrices thereby proving the group isomorphism. This isomorphism immediately implies that the ortho-symplectic group acts transitively on the constant

energy sphere. This is used in the section III.

We are interested in finding the action of this group on the phase space via canonical transformations (symplectic diffeomorphisms). For notational convenience let us group the standard canonical variables together and denote them by ω_μ , $\mu = 1, 2, ..., 2n$, the first n being coordinates and the last n being the momenta. The dual of the symplectic form has components, $\Omega^{\mu\nu}$ given by the $\mu\nu$ th element of the matrix $\mathbf{\Omega}$. With this notation, the PB of functions on the phase space and the infinitesimal canonical transformations are given by,

$$\begin{aligned} \{F(\omega)\ ,\ G(\omega)\} &= \Omega^{\mu\nu}\ \partial_\mu F\ \partial_\nu G \\ \delta_\epsilon \omega^\mu &= \epsilon\ \Omega^{\mu\nu}\ \partial_\nu F(\omega) \end{aligned} \tag{28}$$

Functions purely quadratic in ω generate linear canonical transformations and are also closed under the PB's and those which leave the Hamiltonian invariant give the symmetry transformations. Explicitly,

$$\begin{array}{lcl} F & \equiv & \frac{1}{2}\bar{\omega}\ \mathbf{A}\ \omega \qquad ,\quad G\ \equiv\ \frac{1}{2}\bar{\omega}\ \mathbf{B}\ \omega \quad \Rightarrow \\ \{F, G\} & = & \frac{1}{2}\bar{\omega}\ (\mathbf{A\Omega B} - \mathbf{B\Omega A})\ \omega \end{array} \tag{29}$$

The Hamiltonian corresponds to $\mathbf{A} = \mathbf{I}_{2n}$. The Poisson bracket of any G with H vanishes provided the matrix $\mathbf{B}$ commutes with $\mathbf{\Omega}$. Thus generic matrices defining quadratic functions are real, symmetric matrices commuting with $\mathbf{\Omega}$. The matrices $\mathbf{\Omega B}$ are then antisymmetric and provide an isomorphism of quadratic functions to the generators $\mathbf{T}$ of the group $OSp(2n, R)$.

Action of the one parameter group generated by a function G is found from the integral curves of the corresponding Hamiltonian vector field. These curves are defined by the matrix equations, ($\mathbf{T} \equiv \mathbf{\Omega B}$)

$$\frac{d\omega(\sigma)}{d\sigma} = \mathbf{\Omega B}\omega \tag{30}$$

$$\omega(\sigma) = (e^{\sigma\mathbf{\Omega B}})\omega(0) \tag{31}$$

It is convenient to choose a particular grouping of the phase space coordinates and corresponding choice of $\mathbf{\Omega}$. Firstly let us put $n = 2N$, relevant for the present context, so that the phase space is $4N$ dimensional. Arrange the coordinates and momenta as $x_1, y_1, p_{1x}, p_{1y}, ..., x_N, y_N, p_{Nx}, p_{Ny}$ and denote by ω_i the coordinates and momenta of the i^{th} particle. The index i now runs from 1, ..., N and each ω_i is a 4×1 matrix. Correspondingly we choose $\mathbf{\Omega}$ as a block diagonal matrix with N blocks and each block being 4×4 matrix $\mathbf{\Lambda}$. We choose,

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{0}_2 & \mathbf{I}_2 \\ -\mathbf{I}_2 & \mathbf{0}_2 \end{pmatrix} \tag{32}$$

One can choose a basis for $OSp(4N, R)$ as follows. Let us denote the generators as $\mathbf{T}_i$, $i = 1, 2, ..., N$ and $\mathbf{T}_{ij}$ with $i < j$. Each of these are expressed in the block form. The $\mathbf{T}_i$ are block diagonal with a nonzero 4×4 matrix $\mathbf{u}_i$ as the i^{th} block element. The $\mathbf{T}_{ij}$ have a matrix $\mathbf{v}_{ij}$ at the i^{th} row and j^{th} column ($(ij)^{th}$ block) and $-\bar{\mathbf{v}}_{ij}$ at the $(ji)^{th}$ block. In equations,

$$\begin{aligned} (\mathbf{T}_i)_{mn} &= \mathbf{u}_i \; \delta_{im} \; \delta_{mn} \\ (\mathbf{T}_{ij})_{mn} &= \mathbf{v}_{ij} \; \delta_{im}\delta_{jn} - \; \bar{\mathbf{v}}_{ij} \; \delta_{in}\delta_{jm} \end{aligned} \tag{33}$$

With these definitions it is easy to translate the conditions on the generators in terms of the 4 matrices $\mathbf{u}$, $\mathbf{v}$ as:

$$\bar{\mathbf{u}}_i = -\mathbf{u}_i \qquad \mathbf{u}_{ij}\mathbf{\Lambda} = \mathbf{\Lambda}\mathbf{u}_{ij} \;\; ; \qquad \mathbf{v}_{ij}\mathbf{\Lambda} = \mathbf{\Lambda}\mathbf{v}_{ij} \tag{34}$$

Thus the $\mathbf{u}$'s generate $OSp(4, R)$ while the $\mathbf{v}$'s are required to commute with $\mathbf{\Lambda}$. The number of independent $\mathbf{u}$'s is 4 while number of independent $\mathbf{v}$'s is 8 of which 4 are

'diagonal' and 4 are 'off-diagonal'. The dimension of $OSp(4N,R)$ is thus $4N + 8N(N-1)/2 = 4N^2$. These independent matrices can be explicitly chosen in terms of 2×2 Pauli matrices and the identity matrix as:

$$\mathbf{u}_{(1,2,3,4)} \sim \begin{pmatrix} i\sigma_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & i\sigma_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & \mathbf{I}_2 \\ -\mathbf{I}_2 & \mathbf{0}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & \sigma_1 \\ -\sigma_1 & \mathbf{0}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & \sigma_3 \\ -\sigma_3 & \mathbf{0}_2 \end{pmatrix};$$

$$\mathbf{v}_{(1,2,3,4)} \sim \begin{pmatrix} \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 & \mathbf{0}_2 \\ \mathbf{0}_2 & \sigma_1 \end{pmatrix}, \begin{pmatrix} i\sigma_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & i\sigma_2 \end{pmatrix}, \begin{pmatrix} \sigma_3 & \mathbf{0}_2 \\ \mathbf{0}_2 & \sigma_3 \end{pmatrix}, \tag{35}$$

$$\mathbf{v}_{(5,6,7,8)} \sim \begin{pmatrix} \mathbf{0}_2 & \mathbf{I}_2 \\ -\mathbf{I}_2 & \mathbf{0}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & \sigma_1 \\ -\sigma_1 & \mathbf{0}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & i\sigma_2 \\ -i\sigma_2 & \mathbf{0}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & \sigma_3 \\ -\sigma_3 & \mathbf{0}_2 \end{pmatrix}, \tag{36}$$

We are interested in one parameter groups generated by some element $\mathbf{T}$ of the Lie algebra. If $\mathbf{T}$ satisfies: $\mathbf{T}^2 = -\mathbf{P}$ with $\mathbf{P}$ satisfying $\mathbf{P}^2 = \mathbf{P}$, $[\mathbf{T}, \mathbf{P}] = 0$, then it follows that,

$$e^{\sigma \mathbf{T}} = \mathbf{I}_{4N} - \mathbf{P} + \mathbf{P}\left[cos(\sigma) + sin(\sigma)\mathbf{T}\right]\mathbf{P} \tag{37}$$

For the basis generators $\mathbf{T}_i$ we have $\mathbf{P}$ to be block diagonal with $\mathbf{I}_4$ as the i^{th} block while for $\mathbf{T}_{ij}$ we have $\mathbf{P}$ to be the block diagonal matrix with $\mathbf{I}_4$ as the i^{th} and the j^{th} blocks. Using these, the exponentials of basis generators can be evaluated to get the integral curves as, (with obvious notation)

$$\begin{aligned} \omega(\sigma) &= [\,\mathbf{I}'_i \,+\, \mathbf{I}_i \,\{\, cos(\sigma) \,+\, sin(\sigma)\,\mathbf{T}_i \,\}\, \mathbf{I}_i \,]\,\omega(0) \quad \text{and,} \\ \omega(\sigma) &= \left[\,\mathbf{I}'_{ij} \,+\, \mathbf{I}_{ij} \,\{\, cos(\sigma) \,+\, sin(\sigma)\,\mathbf{T}_{ij} \,\}\, \mathbf{I}_{ij} \,\right]\omega(0) \end{aligned} \tag{38}$$

Therefore integral curves of the basis generators are periodic curves. Further, the $\mathbf{T}_i$'s affect only the i^{th} particle position and momenta while the $\mathbf{T}_{ij}$ mix the i^{th} and j^{th}

particles only.

We need to study how the angular momenta J_i and J_{ij} vary under the action of one parameter subgroups. Define the (block) matrices,

$$
\begin{aligned}
(\mathbf{L}_i)_{mn} &\equiv \mathbf{L}\delta_{im}\delta_{mn} \qquad \mathbf{L} \equiv \mathbf{v}_7 \\
(\mathbf{L}_{ij})_{mn} &\equiv \mathbf{L}\{\ \delta_{mn}(\delta_{im}+\delta_{jm}) - \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\ \}
\end{aligned}
\tag{39}
$$

In terms of these matrices the angular momenta are given by:

$$
\begin{aligned}
J_i &= \frac{1}{2}\,\bar{\omega}\ \mathbf{L}_i\ \omega \qquad = \frac{1}{2}\,\bar{\omega}_i\ \mathbf{L}\ \omega_i \\
J_{ij} &= \frac{1}{2}\,\bar{\omega}\ \mathbf{L}_{ij}\ \omega \qquad = \frac{1}{2}\,\bar{\omega}_{ij}\ \mathbf{L}\ \omega_{ij}\ , \qquad \omega_{ij} \equiv \omega_i - \omega_j.
\end{aligned}
\tag{40}
$$

Under the group generated by the basis generator, $\mathbf{T}_i$, the J_m for instance varies as:

$$
\begin{aligned}
2J_m(\sigma) = 2J_m + \delta_{im}\Big[&-sin^2(\sigma)\ \{2J_i - \bar{\omega}_\mathbf{i}\ (\bar{\mathbf{u}}\ \mathbf{L}\ \mathbf{u})\ \omega_\mathbf{i}\} \\
&+ sin(\sigma)cos(\sigma)\ \{\ \bar{\omega}_\mathbf{i}\ (\bar{\mathbf{u}}\mathbf{L} + \mathbf{L}\mathbf{u})\ \omega_\mathbf{i}\ \}\ \Big]
\end{aligned}
\tag{41}
$$

while under the group generated by the basis generator, $\mathbf{T}_{ij}$, the J_m varies as:

$$
\begin{aligned}
2J_m(\sigma) = 2J_m + \delta_{im}\Big[&-sin^2(\sigma)\ \{2J_i - \bar{\omega}_\mathbf{j}\ (\bar{\mathbf{v}}\ \mathbf{L}\ \mathbf{v})\ \omega_\mathbf{j}\} \\
&+ 2sin(\sigma)cos(\sigma)\ \{\ \bar{\omega}_\mathbf{i}\ (\mathbf{L}\mathbf{v})\ \omega_\mathbf{j}\ \}\ \Big] \\
\delta_{jm}\Big[&-sin^2(\sigma)\ \{2J_j - \bar{\omega}_\mathbf{i}\ (\mathbf{v}\ \mathbf{L}\ \bar{\mathbf{v}})\ \omega_\mathbf{i}\} \\
&- 2sin(\sigma)cos(\sigma)\ \{\ \bar{\omega}_\mathbf{i}\ (\mathbf{v}\mathbf{L})\ \omega_\mathbf{j}\ \}\ \Big]
\end{aligned}
\tag{42}
$$

Similar but more complicated expressions follow for the $J_{mn}(\sigma)$ also.

Remark: For the case of $N = 1$ we have only one block. This could be either a single two dimensional oscillator OR the relative coordinate dynamics of two anyons.

The generators are of course only $\mathbf{u}$'s. Of these $\mathbf{u}_2$ corresponds to the Hamiltonian itself while $\mathbf{u}_1$ generates same rotations of both $\vec{r}, \vec{p}$. These two matrices commute with $\mathbf{L}$ while the remaining two anti-commute with $\mathbf{L}$. This leads to the result quoted in the section III .

To deduce the surviving symmetry group for many anyons we need infinitesimal variations of J_{mn} for arbitrary generator $\mathbf{T}$. This is easily derived and is given by,

$$\delta J_{mn} \;=\; \sum_j \bar{\omega}_{\mathbf{mn}}\ \mathbf{L}\ (\mathbf{T}_{mj} - \mathbf{T}_{nj}\ )\omega_j \tag{43}$$

In section III we needed to determine $\mathbf{T}$ such that δJ_{mn} is zero *whenever* J_{mn} is zero. That is, at all points in the phase space where any single J_{mn} is zero, we want its infinitesimal variation induced by $\mathbf{T}$ to be zero.

Fix a particular m and n. Fix ω_{mn}. We can consider points with $\omega_j,\ j \neq m, n$ such that no other J_{ij} is zero. $\delta J_{mn} \;=\; 0$ then implies $\mathbf{T}_{mj} \;=\; \mathbf{T}_{nj}\ \forall\ j \neq m, n$. We can also consider points with different ω_m, ω_n keeping ω_{mn} fixed and maintaining all other conditions. This implies $\mathbf{T}_{mm} - \mathbf{T}_{nm} + \mathbf{T}_{mn} - \mathbf{T}_{nn} = 0$. Repeating this for all $m \neq n$ fixes the form of $\mathbf{T}$ in terms of arbitrary generators of $OSp(4,R)$, $\mathbf{u}$, $\mathbf{v}$ as,

$$(\mathbf{T})_{ij} \;=\; \mathbf{u}\delta_{ij} \;+\; \mathbf{v}(1-\delta_{ij}) \quad , \quad [\ \mathbf{L}\ ,\ \mathbf{u}-\mathbf{v}\ ] \;=\; 0 \tag{44}$$

The choice $\mathbf{u} = \mathbf{v}$ corresponds to $\mathbf{T}_{ij} \;=\; \mathbf{u}\ \forall\ i, j$. It follows that *all* J_{mn} 's are invariant independent of their values. It is easy to see that these *four* $\mathbf{T}$'s effect transformations of the center-of-mass variables which are insensitive to the anyonic features. This $OSp(4,R)$ symmetry is thus always present for all $N \geq 2$.

The choice $\mathbf{v} = 0$ implies $\mathbf{T}$ is block diagonal with the same $\mathbf{u}$ on all the diagonal blocks. Further, $[\mathbf{L}, \mathbf{u}] = 0$ implies that $\mathbf{u}$ must be a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. These *two* $\mathbf{T}$'s can be seen to correspond to the total Hamiltonian and the total angular momentum. This result is used in the section III to deduce that the surviving symmetry for N anyons is $OSp(4, R) \times O(2, R) \times O(2, R)$.

Appendix B : **Regularised classical dynamics and reflecting orbits**

Consider without loss of generality the case of two anyon in the relative coordinates. Generic orbits in the configure space, $R^2 - \{\vec{0}\}$, are of course an ellipse and only a degenerate elliptical orbit attempts to pass through the origin. Noting that this system can be thought of as a charged particle in presence of a singular ("statistical") magnetic field at the origin, one may regularise the magnetic field or the flux to study the orbits and obtain the limiting behaviour to deduce "boundary condition" at the origin.

To do this we imagine the relative Hamiltonian to be that of a particle in a magnetic field along the z axis. The field is of course to be effectively confined to a small disc around the origin. Now observe that for an axially symmetric magnetic field along the z-axis, $B(\vec{r}, \theta) = B(r)\hat{k}$ where r, θ are the usual spherical coordinates in two dimensions, we may write the vector potential as,

$$\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} \tag{45}$$

This implies,

$$B(r) = \partial_r A_\theta + \frac{A_\theta}{r} - \frac{1}{r}\partial_\theta A_r, \tag{46}$$

In a symmetric gauge, the vector potential is independent of the polar angle and by choosing it to be divergence free one can set the radial component to zero.

The flux $\Phi(R)$, through a disc of radius R is given by,

$$\Phi(R) \; = \; 2\pi \int_0^R dr r B(r) \quad = R \oint A_\theta d\theta \; = \; 2\pi R A_\theta(R). \tag{47}$$

This implies,

$$A_\theta(r) = \frac{\Phi(r)}{2\pi r} = \frac{1}{r}\int_0^r dr' r' B(r') \tag{48}$$

The choice, $\Phi(r) = 2\pi\alpha \ \forall \ r > 0$, gives the vector potential used in the quantum mechanical calculation. It also implies the $rB(r) = \alpha\delta(r)$ and this of course is the singular nature of the magnetic field. Notice that smearing the $\delta(r)$ will not make the magnetic field non-singular at the origin because of the explicit $1/r$.

For a regulated system one wants the magnetic field to be non-singular every where and effectively confined to a disk of radius ϵ around the origin. This requires the vector potential A_θ also to be non-singular and therefore vanishing at the origin. If $A_\theta(r) \rightarrow cr^\beta$ as r approaches zero then, $B(r) \rightarrow c(\beta+1)r^{\beta-1}$. For a finite, nonzero $B(0)$ we must have $\beta = 1$ and therefore $\Phi(r) \rightarrow 2\pi c r^2$ near the origin. For $r \geq \epsilon$ we still retain the flux to be $2\pi\alpha$. Continuity at $r = \epsilon$ then gives $c = \alpha/\epsilon^2$. Notice that this limiting behaviour is fixed by the demand of non-singularity of the fields and as such must be reflected in any explicit choice for the magnetic field.

Thus our regulation involves choosing,

$$\begin{aligned} \Phi(r) &= 2\pi\alpha \qquad \forall \ r \geq \epsilon \\ &\rightarrow 2\pi\alpha\frac{r^2}{\epsilon^2} \quad \text{as} \ \ r \ \rightarrow 0 \end{aligned} \tag{49}$$

One could choose a uniform nonzero magnetic field inside the disk as an explicit choice but it will not be necessary. Since we are interested in the limiting behaviour of trajectories as ϵ is taken to zero, the limiting behaviour of the flux is all that we need.

Consider now the orbit equation for ϵ nonzero. Integrating the equations of motion once using the two constants, energy E and angular momentum ℓ, the orbit equations

in r, θ coordinates become :

$$\begin{aligned}
\dot{r} &= \pm\sqrt{2E - r^2\dot{\theta}^2 - r^2} && \\
r^2\dot{\theta} &= \ell - rA_\theta(r) && \qquad (50)\\
&= \ell - \alpha\frac{r^2}{\epsilon^2} && r < \epsilon \\
&= \ell - \alpha && r \geq \epsilon
\end{aligned}$$

There are three types of orbits possible: those which are fully inside the disc, those which are fully outside the disc and those which go both inside and outside the disc. In the limit of ϵ going to zero, the first type of orbits are clearly irrelevant. It is easy to see that for the second type of orbits one must have $\ell \neq \alpha$. These are insensitive to the flux in the limit and are thus identical to the orbits of the oscillator.

For the last type, an interior turning point is possible only for $0 \leq \ell \leq \alpha$. The condition that such an orbit must also have an exterior turning point limits ℓ to α. We are interested in computing the change in the angular coordinate from the entry into the disk till exit from it. Explicit computation shows that the change in the angular coordinate goes to zero as ϵ goes to zero. Thus such an orbit *reflects* at the origin. Note that these are precisely the radial orbits.

To summarize, a regulated classical modeling for anyons is generically stipulated by giving the behaviour of the flux near the origin. It amounts to cutting out a disk of radius ϵ and filling it up with non-singular fields. The classical non-radial orbits then are exactly same as those of the oscillator except for the replacement $l \to l - \alpha$. The radial orbits reflect at the the origin and are termed *half orbits* since their period is half of that for the other orbits.

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